Vassiliev Invariants and Gleam Polynomials

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Abstract

It was shown by Goussarov [G1] that Vassiliev invariants are polynomials in the gleams for a fixed Turaev shadow. In this paper we show that Vassiliev invariants are almost characterized by this fact. We also prove that the space of knot invariants which are polynomials in the gleams is bigger than the Vassiliev subspace.

Introduction

In Turaev's shadow topology [Tu] a knot in S^3 is presented by its Turaev shadow, which is its image under the Hopf projection, and its gleams, which are integer numbers associated to the regions which enable us to reconstruct the knot up to isotopy. It is a very interesting question how knot invariants depend on the gleams for a fixed Turaev shadow. Studying this question or searching for explicit formulas it is easier to work with the parametrization $\mathcal{K}$ introduced in [B], than with the gleams directly. This map $\mathcal{K}$ is defined on a lattice $\mathbb{Z}^e \times \mathbb{Z}$, where e is the number of double points of the fixed Turaev shadow. One of the first results was the following: Let us define Jones' Vassiliev invariants $\{u_n\}_{n\geq 2}$ via the Jones polynomial $J_t(K)$, by $J_{e^x}(K) = \sum_{n=0}^{\infty} u_n(K)x^n$.

Theorem 0.1 [B] *The function $u_n \circ \mathcal{K} : \mathbb{Z}^e \times \mathbb{Z} \longrightarrow \mathbb{Q}$ is a polynomial of degree $2n$.*

Before we state the main results, let us recall the following. A knot invariant v with values in some abelian group $\mathcal{G}$ is a polynomial of degree m in the gleams, if and only if $v \circ \mathcal{K}$ is a polynomial of degree m.

1   The Main Results

In the following we will state all results for knots only, although most of them easily extend to links. Let v_n be a Vassiliev invariant of order n with values in $\mathcal{G}$.

Theorem 1.1 (Goussarov [G1]) *The function $v_n \circ \mathcal{K} : \mathbb{Z}^e \times \mathbb{Z} \longrightarrow \mathcal{G}$ is a polynomial of degree $\leq 2n$.*

Remark Goussarov proved that derivatives of order $2n + 1$ vanish. However, he cannot give explicit formulas for $v_n \circ \mathcal{K}$.

Theorem 1.2 *Let v be a knot invariant with values in $\mathcal{G}$. Assume that for every fixed Turaev shadow v is a polynomial in the gleams of degree $\leq m$. Then v is a Vassiliev invariant of order $\leq m$.*

Remark The main point (due to Goussarov [G2]) is that one should not try to prove that a knot invariant which is a polynomial in the gleams of degree $\leq 2m$ is a Vassiliev invariant of order $\leq m$.

Proof. We have to show that v vanishes after the usual extension to singular knots on every $(m+1)$-singular knot $K^{(m+1)}$. Suppose that $K^{(m+1)}$ is given by a standard diagram on $\mathbb{R}^2$. We would like to apply Thereom 3.2 in [B] where alternating sums of values of a knot invariant v are related with partial derivatives (finite differences) of $v \circ \mathcal{K}$. First, we fix the Turaev shadow s on $\mathbb{R}^2 \cup \{\infty\}$ given by the projection of $K^{(m+1)}$. We enumerate the double points of s. Let $P_1, ..., P_{m+1}$ be the double points of s coming from the $m+1$ singular points of $K^{(m+1)}$ and $P_{m+2}, ..., P_e$ be the remaining double points of s coming from the negative and positive crossings of $K^{(m+1)}$. Then we know that

$$v\left(K^{(m+1)}\right) = \Delta_{x_1} \Delta_{x_2} ... \Delta_{x_{m+1}} (v \circ \mathcal{K})(0, 0, ..., 0, \beta_{m+2}, ..., \beta_e, 0) \ ,$$

where β_k is 0 or 1 if the crossing in P_k of $K^{(m+1)}$ is negative or positive, respectively. However, by assumption, $v \circ \mathcal{K}$ is a polynomial of degree $\leq m$, so the $m + 1$ partial derivatives of $v \circ \mathcal{K}$ vanish and we are done. $\qquad\square$

Theorem 1.3 *There exist $\mathbb{C}$-valued knot invariants, which are polynomials in the gleams for every fixed Turaev shadow, but are not Vassiliev.*

Remark The point here is that the degree of these polynomials is not uniformly bounded, when we change the Turaev shadows. The idea of the following construction is due to Goussarov [G2].

Proof. We use the fact, that the dimensions $\dim_{\mathbb{C}}(V_n)$ of the vector spaces of Vassiliev invariants of order $\leq n$ grow faster than polynomially, when n tends to $+\infty$, see [K]. Kontsevich states that

$$\dim_{\mathbb{C}}(V_n) > e^{c\sqrt{n}} \ , \qquad n \to +\infty \ ,$$

for any positive constant $c < \pi\sqrt{2/3}$. Let us denote by m_k the number of Turaev shadows with $\leq k$ regions, and by $W_{2n,k}$ the vector space of polynomials (not necessarily knot invariants) of degree $\leq 2n$ in $\leq k$ variables. We define by $\pi_{n,k}$ the linear map which associates to any Vassiliev invariant of order $\leq n$, the m_k polynomials of order $\leq 2n$ in $\leq k$ variables, see Theorem 1.1.

$$\pi_{n,k} : V_n \longrightarrow (W_{2n,k})^{\oplus m_k} \ .$$

The dimension of V_n/V_{n-1} grows faster in n than $\dim(W_{2n,k})^{\oplus m_k}$ for any fixed k; the first growth is subexponential, the second one polynomial. Therefore there exists for any k a Vassiliev invariant $v_{n_k} \in V_{n_k} \setminus V_{n_k-1}$ with $\pi_{n_k,k}(v_{n_k}) = 0 \in (W_{2n_k,k})^{\oplus m_k}$. This means that v_{n_k} is a Vassiliev invariant of order n_k (not of any lower order), which vanishes on all knots which admit a Hopf projection with $\leq k$ regions. Now define

$$v := \sum_{k=0}^{\infty} v_{n_k}.$$

Note that v is well defined because for every knot the sum is finite. $\qquad\square$

Remark This result shows that the space of knot invariants which are polynomials in the gleams is bigger than the Vassiliev subspace. It is known that many "classical" knot invariants are not Vassiliev invariants but unfortunately they are not polynomials in the gleams either, as we will see in the next theorem. Let us mention one more thing first. In [Tr] it was shown that the uniform limit of Vassiliev invariants is again Vassiliev. Using the fact that these Vassiliev invariants give rise to polynomials defined on the lattice $\mathbb{Z}^e \times \mathbb{Z}$ it is easy to check that such a sequence $(v^{(n)})_{n \in \mathbb{N}}$ has to stabilize, if it is normalized, let us say by requiring $v^{(n)}(\text{unknot}) = 0$. The same is true for a uniform convergent sequence of knot invariants, which are polynomials in the gleams.

Theorem 1.4 *The following knot invariants are not polynomials in the gleams: the unknotting number, genus, signature, bridge number, braid index, span of the Jones polynomial.*

Proof. It is enough to study the behaviour of these invariants on the series of knots denoted by $\mathcal{K}(x_f)$, see [B]. These are the only knots which admit a Hopf projection without any double points. Recall that for $x_f \geq 2$, $\mathcal{K}(x_f) =$ torus knot$(x_f, x_f + 1)$ and $\mathcal{K}(x_f)$ is trivial for $x_f \in \{-2, -1, 0, 1\}$. All the knot invariants mentioned above have the same behaviour: They are not constant on $\mathcal{K}(x_f)$ but they grow only linearly or at most quadratically in x_f, and therefore they are not polynomials in x_f. $\qquad\square$

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